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CFT/CFT interpolating RG flows and the holographic c-function

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Abstract: We consider holographic RG flows which interpolate between non-trivial ultra-violet (UV) and infra-red (IR) conformal fixed points. We study the "superpotentials" which characterize different flows and discuss their expansions near the fixed points. Then we focus on the holographic c-function as defined from the two-point function of the stress-energy tensor. We point out that the equation for the metric fluctuations in the background flow is equivalent to a scattering problem and we use this to obtain an expression for the c-function in terms of the associated Jost functions. We propose two explicit models that realize UV-IR flows. In the first example we consider a thin wall separating two AdS spaces with different radii, while in the second one the UV region is replaced with an asymptotically AdS space. We find that the holographic c-function obeys the expected properties. In particular it reduces to the correct — non zero — central charge in the IR limit.

Keywords: Renormalization Regularization and Renormalons, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Since the work of Zamolodchikov [1], the proof of a c-theorem for quantum field theories in more than two dimensions has remained an open problem. In the last years the idea of holography, namely the extension of the AdS/CFT correspondence to non conformal theories, has provided new tools for addressing this issue. In this context, it was shown [2, 3] that in any dimension there exists a function of the gravity fields which, under suitable physical assumptions, fulfills the basic requirements of a central function, i.e. a monotonic function of some energy scale, which in the IR and UV limits reduces to the central charges of the fixed point conformal field theories [4]. Since it is not obvious how observables on the field theory side are related to the supergravity fields, the field theoretical meaning of this function has remained unclear. In [5] a natural definition of a holographic c-function was proposed, which is related to the OPE of the stress-energy tensor [4] as (here in $d=4$)

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = -\frac{1}{48\pi^4}\Pi^{(2)}_{\mu\nu\rho\sigma}\left[\frac{c(x)}{x^4}\right] + \pi_{\mu\nu}\pi_{\rho\sigma}\left[\frac{f(x)}{x^4}\right], \tag{1.1}$$

where $\pi_{\mu\nu} = \partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2$ and $\Pi^{(2)}_{\mu\nu\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - 3(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$. Hence one can compute it in holographic flows by analyzing the fluctuations of the transverse traceless (TT) graviton in a given background [6, 7]. This proposal has been tested in a few cases where the flows are known explicitly, and for which the fluctuations can be expressed in closed form [8, 9]. However, these flows are all singular in the IR, describing non conformal theories and/or vacua (so-called Coulomb branch flows). Moreover, a proof that the suggested c-function satisfies its characterizing requirements is still missing.

In order to study the holographic central function further, one should construct explicitly background RG flows, which interpolate between two non-trivial fixed point conformal field theories. By "interpolating" flows we mean domain wall solutions (not necessarily supersymmetric), which start at a critical point of the supergravity potential and end at a nearby critical point. Even though there are a few cases in which such flows are known implicitly [2, 10, 3], it is somewhat disappointing that no analytic solutions have been found so far.

A useful method for finding supergravity solutions is to rewrite the equations of motion as a first order system, supplemented with a non-linear equation for a "superpotential" (see e.g. [11, 12]). Experience has shown that when such a "superpotential" can be written down explicitly this is an unmistakable sign of underlying supersymmetry. Nevertheless, one is not committed to restrict to such a class of solutions, and in fact the possible solutions that arise from a given supergravity potential are far more richer [12]. It turns out that there are infinite solutions which represent deformations of a UV conformal field theory. Two of them are somewhat special and correspond to flows dual to operators acquiring a vacuum expectation value (Coulomb branch flows) and the interpolating flow. In this paper we study global and local properties of the "superpotential", in order to characterize the interpolating solutions.

The problem of solving the TT graviton fluctuation in a Poincaré invariant flow leads generically to a Schrödinger equation with a potential of supersymmetric type [13] which depends non trivially on the original supergravity potential. We point out that for an interpolating RG flow the problem is that of a partial wave scattering off a central potential. This allows to study properties of the fluctuations without detailed knowledge of the flow and to obtain some informations about the associated holographic c-function. A general proof that the c-function satisfies the expected properties still remains an open problem. Nevertheless, we have checked that these properties are respected in some simplified models, where the background flow is obtained by patching together space-times on which the fluctuation can be solved in closed form. Some properties of two-point functions in generic interpolating flows were studied in [14]. Previous studies of Schrödinger potentials associated with the TT graviton fluctuations appeared for instance in [13, 15, 16]. However, in these cases, the analysis was devoted to IR-singular supersymmetric flows.

The paper is organized as follows. In section 2 we give a detailed analysis of solutions related to a given potential in a one-scalar model of the holographic RG. We study the global structure of the "superpotential" and write out the local expansion of some particular solutions near the fixed points. In section 3 we move to the analysis of the fluctuations and the associated quantum mechanical potential. We write down an expression for the holographic c-function in terms of Jost functions of a suitable scattering problem, and obtain some general properties consistent with field theory expectations. Finally, in section 4, we study the c-functions related to two examples which model the flows with piecewise simple space-times. This allows us to test the general prescription for computing correlation functions in IR-conformal flows.

2. Background flows

We begin with a review of holographic RG flows, explaining in particular how different trajectories arise from a given supergravity potential. The starting point is an action in $d+1$ dimensions[1]

$$S = \int d^{d+1}x\sqrt{g}\Big(R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big) - 2\int d^dx\sqrt{\hat{g}}K \tag{2.1}$$

which one can think of as a truncation of some supergravity action. We consider the Poincaré invariant ansatz for the background flow and work from the beginning in the conformal gauge

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{2A(z)}(\mathrm{d}z^2 + \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \\ \phi &= \phi(z)\,. \end{aligned} \tag{2.2}$$

We want to address trajectories that interpolate between two AdS fixed points, therefore we shall assume that the potential possesses two nearby extrema, and so V must be negative definite. The second order equations of motion resulting from (2.1) are equivalent to the following first order equations

$$\frac{dA}{dz} = \frac{\mathrm{e}^A}{2(d-1)}W \qquad \frac{d\phi}{dz} = -\,\mathrm{e}^A\,W_\phi\,, \tag{2.3}$$

with the "superpotential" W defined by the equation

$$V = \frac{1}{2}W_\phi^2 \;-\; \frac{d}{4(d-1)}W^2\,. \tag{2.4}$$

For any given solution of (2.4) there is a mirror one, which has opposite overall sign. We will restrict the analysis to those which are negative definite.

[1]Our convention for the metric signature is $\eta_{ij} = (-,+,\cdots,+)$. And we have set the Newton constant $\kappa = 1$.

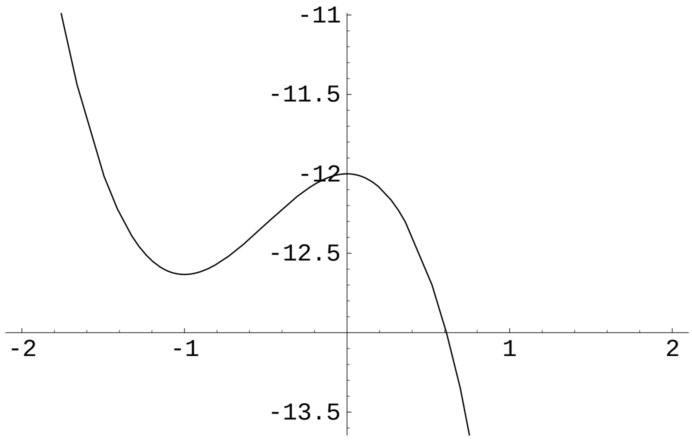

Figure 1: The cubic potential $V = -12 + 1/2m^2\phi^2 + 1/3m^2\phi^3$ with $m^2 = -3.8$.

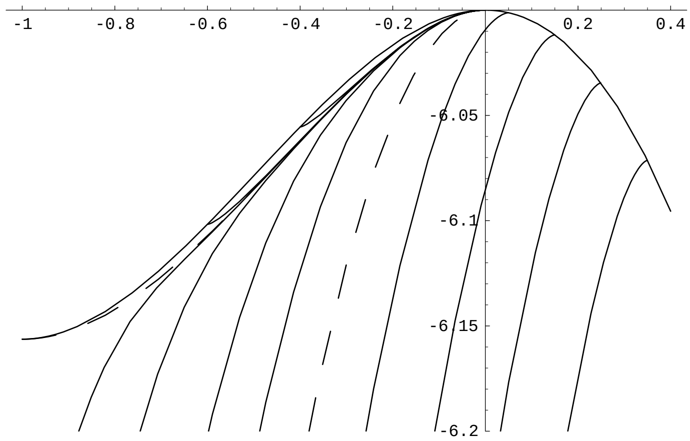

Figure 2: W. Solutions corresponding to the potential of figure 1. The dashed curves correspond to the interpolating and "Coulomb branch" solutions.

2.1 Solutions for the superpotentials

The non-linear nature of (2.4) makes the analysis of possible trajectories in the phase space (W-ϕ plane) non-trivial. In spite of being a first order equation, it is not true that in general a one-parameter family of solutions covers the full phase space. In this respect, the existence of *supersymmetric* solutions is a remarkable example of isolated trajectories. These solutions arise, generically, by requiring vanishing dilatino and gravitino variations, and are therefore the true superpotentials of the theory.

In order to discuss the different kind of solutions we have solved numerically (2.4) for some potential which posses one maximum (UV fixed point) and one minimum (IR fixed point). In figure 1 it is depicted a typical potential.

As shown in figure 2, the phase space is bounded from above by the curve $V(\phi) + \frac{d}{4(d-1)}W_{\rm up}^2 = 0$. The two critical points of V correspond to critical points in the phase space and it turns out that the IR one is repulsive, while the UV one is attractive. There is a continuum of solutions which approach the UV fixed point, whereas only one originates from the IR fixed point.

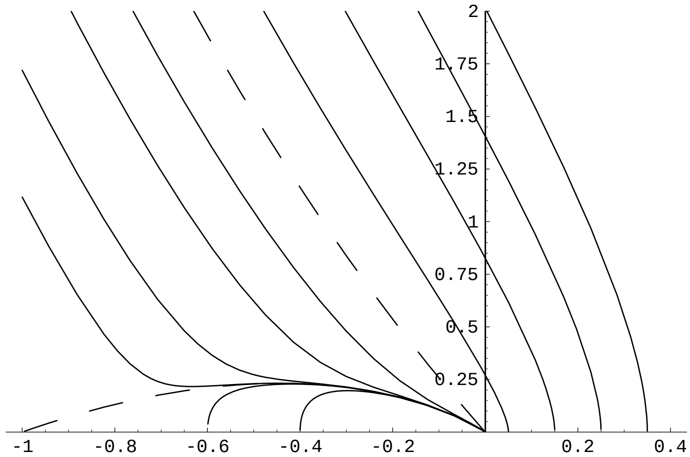

Figure 3: W_ϕ. Solutions corresponding to the potential of figure 1. The dashed curves correspond to the interpolating and "Coulomb branch" solutions.

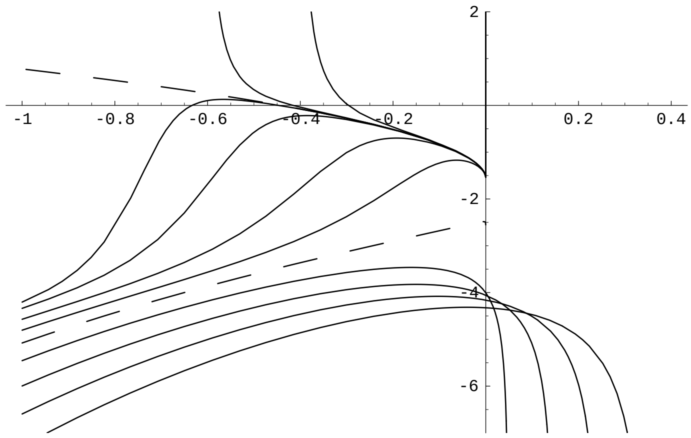

Figure 4: $W_{\phi\phi}$. Solutions corresponding to the potential of figure 1. The dashed curves correspond to the interpolating and "Coulomb branch" solutions.

There are two distinguished curves (dashed lines in the figures): one is the interpolating trajectory, the other a lower bound delimiting the solutions flowing to the UV point. The distinguished role played by these two solutions is better understood by looking at the first and second derivatives, shown in figure 3 and 4. In particular, notice that for all the solutions ending at the UV fixed point (including the interpolating one) the second derivative $W_0^{(2)} = \Delta - d \approx -1.55$ at the UV point, whereas there is only one solution such that $W_0^{(2)} = -\Delta \approx -2.45$. Δ is the conformal dimension of the dual CFT operator, which is related to the mass of the scalar field by the usual formula $\Delta = (d + \sqrt{d^2 + 4m^2R^2})/2$. The latter solution coincides usually with the supersymmetric superpotential when supersymmetry is preserved [12, 17]. Notice that the solutions not flowing to the UV point must end at the upper bound of the allowed phase space and are all singular there, as can be seen from the second derivatives blowing up.

Before going to the discussion of the local expansions near the fixed points, let us remark on the issue of stability. In view of the fact that there exist more solutions flowing to the UV point than only the two found by Taylor expansion (see below), one could ask

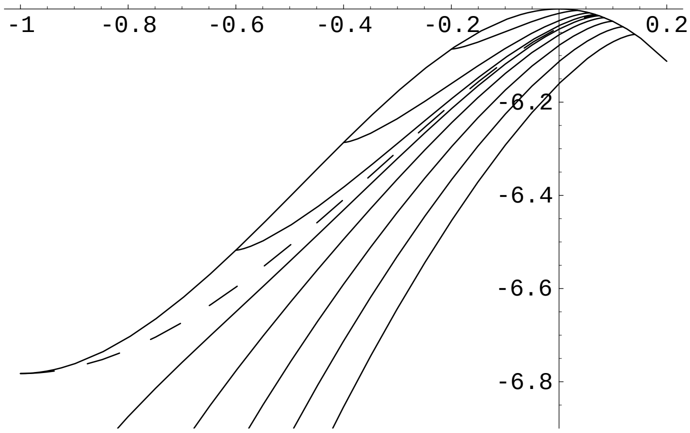

Figure 5: W. Solutions for W for a mass $m^2 = -20$ violating the BF bound. The dashed curves correspond to the solution originating in the IR fixed point.

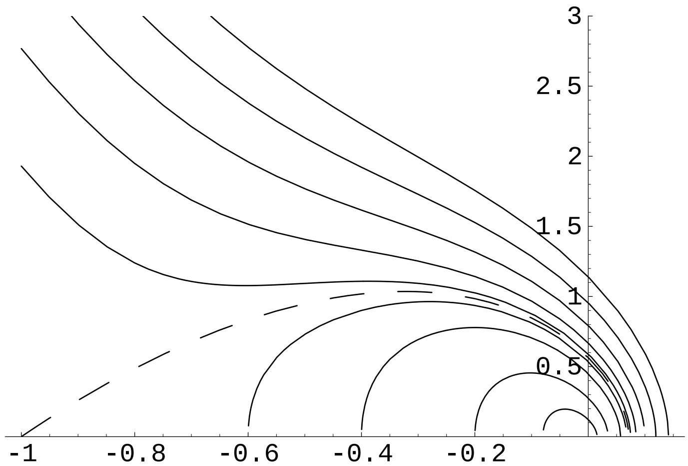

Figure 6: W_ϕ. Solutions for W for a mass $m^2 = -20$ violating the BF bound. The dashed curves correspond to the solution originating in the IR fixed point.

whether some of these solutions do exist even for scalar modes violating the Breitenlohner-Freedman (BF) bound $m^2R^2 < -d^2/4$ [18]. In fact, the numerical solutions show that in this case there are no trajectories flowing to the UV point, as depicted in figure 5. Therefore the condition for stability discussed in [19] is valid beyond the local Taylor expansion.

The situation for a mass precisely at the BF bound is the same as in the BF-stable case. There is still a continuum of solutions ending in the UV fixed point, all of them having the same second derivative $W_0^{(2)} = \Delta - d = -\Delta$.

2.2 Expansion of the superpotentials near the fixed points

We have seen that the interpolating solution should exists in general. We will proceed now with the local analysis of W in the vicinity of the AdS fixed points, which are by definition critical points of the gravity potential V, so elucidating the meaning of the two distinguished solutions. In general we do not need to specify whether we are near an IR or

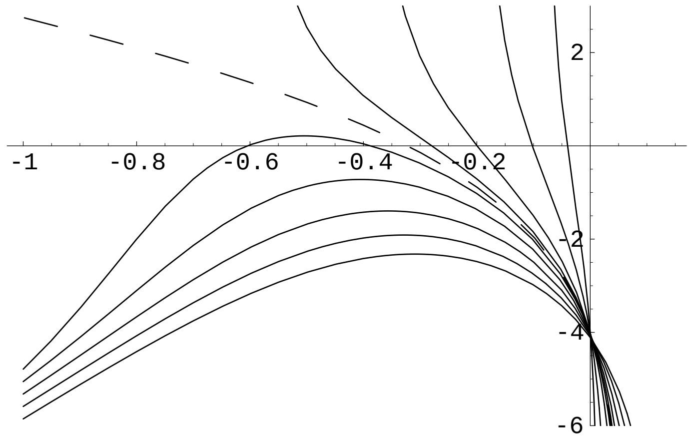

Figure 7: $W_{\phi\phi}$. Solutions for W for a mass $m^2 = -20$ violating the BF bound. The dashed curves correspond to the solution originating in the IR fixed point.

UV fixed point. Near a critical point ϕ_0, we write the gravity potential as

$$V(\Phi) = -\frac{d(d-1)}{R^2} + \frac{1}{2}m^2\Phi^2 + \frac{1}{3!}V^{(3)}\Phi^3 + \cdots, \tag{2.5}$$

where $\Phi = \phi - \phi_0$, and we have expressed the constant term in terms of the AdS radius R. Let us assume that a Taylor expansion for W near ϕ_0 holds, and expand equation (2.4) in power series, using the ansatz

$$W(\Phi) = W_0 + W_0^{(1)}\Phi + \frac{1}{2}W_0^{(2)}\Phi^2 + \frac{1}{3!}W_0^{(3)}\Phi^3 + \cdots. \tag{2.6}$$

If $W_0^{(1)} \neq 0$, the resulting zero-th order equation contains the two undetermined coefficients W_0 and $W_0^{(1)}$, one of which, say W_0, can be taken as an integration constant. Equating the linear terms gives $W_0^{(2)} = d/(2d-2)\ W_0$, showing that $W_0^{(2)}$ is always negative, which for instance is observed in figure 4. Higher order equations are algebraic[2] equations for $W_0^{(n)}$ in terms of the Taylor coefficients of V, which parametrically depend on W_0.

If $W_0^{(1)} = 0$, the expansion of (2.4) is substantially changed, and needs to be analyzed separately. Since in this case a critical point of W does coincide with one of V, we can write the more familiar expansion [20] in terms of the conformal dimension Δ

$$W(\Phi) = -\frac{2(d-1)}{R} + \frac{\Delta - d}{2R}\Phi^2 + \frac{1}{3!}\frac{RV_0^{(3)}}{3\Delta - 2d}\Phi^3 + \cdots. \tag{2.7}$$

Notice that Δ is usually thought of as pertinent to the UV fixed point, in which case $0 > m^2R > -d^2/4$ and the conformal dimension ranges accordingly in

$$d > \Delta_{\rm UV} \geq d/2. \tag{2.8}$$

[2]There is no obstruction in solving these equations, contrary to the $W_0^{(1)} = 0$ case.

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In the IR instead, the potential is at a local minimum, thus $m^2 > 0$, and in this case the conformal dimension has no upper bound, satisfying in general only

$$\Delta_{\rm IR} > d\,. \tag{2.9}$$

The second order in the Taylor expansion is quadratic in $W_0^{(2)}$, therefore there exists another solution which gives rise to a superpotential

$$\hat{W}(\Phi) = -\frac{2(d-1)}{R} - \frac{\Delta}{2R}\Phi^2 - \frac{1}{3!}\frac{RV_0^{(3)}}{3\Delta - d}\Phi^3 + \cdots\,. \tag{2.10}$$

The expansion (2.10) corresponds clearly to a Coulomb branch flow. The interpolating solution instead, corresponds to a true deformation. Some numerical analysis has given evidence that this is actually the solution which admits a Taylor expansion, though its radius of convergence is not known in general. The higher level equations of the expansion take the following form

$$[n\Delta - (n-1)d]\, W_0^{(n)} = RV_0^{(n)} + P(\Delta, V_0^{(3)}, \ldots, V_0^{(n-1)})\,, \tag{2.11}$$

where P is a polynomial in the Taylor coefficients of V. Thus, we find that there is a series of operators with rational dimensions for which the Taylor expansion of W explicitly breaks down. The expansion does not break down if the left hand side of (2.11) vanishes identically. This happens for instance for the flow found in [21], where a supersymmetric (and analytic) superpotential exists, despite the operator has dimension $\Delta = 3$. In general the impossibility of performing a Taylor expansion does not imply that no solution exists. In fact, we find that insertion of a logarithmic term is enough to solve the equation, at least locally. This solution reads

$$W(\Phi) = -\frac{2(d-1)}{R} + \frac{\Delta - d}{2R}\Phi^2 + \cdots + \frac{1}{n!}W_0^{(n)}\Phi^n + \frac{1}{n!}X_0^{(n)}\Phi^n \log\Phi + \cdots\,. \tag{2.12}$$

Higher order terms contain powers of logs and all the coefficients of the series are fixed by the differential equation (2.4), except $W_0^{(n)}$, which remains as a new integration constant. The structure of this expression is formally similar to the expansion that arises in solving the Einstein equations near an AdS fixed point [22]. In that case a logarithmic term appears in the metric at order d, and for the scalar field at order $n = 2\Delta - d$, whereas here we find that it occurs for operators of dimension

$$\Delta = \frac{n-1}{n}\,d\,, \tag{2.13}$$

where n is an integer $n \geq 3$. Both types of operators contribute to the conformal anomaly in the fixed point CFT [23, 20]. We have shifted a discussion of the above expansion and the special conformal dimensions (2.13) to appendix B, since it is not needed for what follows.

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2.3 Expansion of the fields near the fixed points

Now we turn to the first order equations for the fields, and solve them perturbatively, near a fixed point. From (2.3) one gets

$$\frac{dA}{d\phi} = -\frac{1}{2(d-1)}\frac{W}{W_\phi} \tag{2.14}$$

which can be solved for A upon expanding the right hand side near the fixed point. Afterwards, integrating order by order, one finds a series for z in terms of ϕ. This series can be inverted, the leading term being linear, so that eventually one can read off the asymptotic expansions for $A(z)$ and $\phi(z)$. It turns out that the following expansions hold

$$\phi(z) = \phi_0 + \tau\left[1 + c_1\tau + c_2\tau^2 + \mathcal{O}(\tau^3)\right] \tag{2.15}$$

$$\mathrm{e}^{A(z)} = \frac{R}{z-z_0}\left[1 + a_2\tau^2 + a_3\tau^3 + \mathcal{O}(\tau^4)\right], \tag{2.16}$$

where

$$\tau \equiv \left(\frac{z-z_0}{R\mathrm{e}^{-A_0}}\right)^{d-\Delta}. \tag{2.17}$$

The first few coefficients of the expansions can be found in Appendix C.[3] The expansions are sensible both in the UV and IR limits, as the sign of $d-\Delta$ flips appropriately in the two cases. The two integration constants appear in a substantially trivial way: z_0 is a shift of z and A_0 a rescaling. We will omit them in the following, with the proviso that one should not prescribe them simultaneously in UV and IR regions. In the UV a similar expansion holds using the hatted superpotential (2.10), with the change that the exponent in (2.17) is now Δ.

From the above expansions we can confirm that $\hat{W}$ produces a Coulomb branch flow, as the leading behavior of the scalar field for small z is $\Phi(z) \sim z^{\Delta}$. Conversely, for a superpotential of type (2.7), the leading behavior of (2.15) is $\Phi(z) \sim z^{d-\Delta}$, which is a deformation flow. Moreover, a closer look to (2.15) also shows that at subleading order the signature of a vev never appears. In fact this would be the case if for some integer n, $(d-\Delta)n = \Delta$, but we have excluded these cases from the analysis, in that they led to a superpotential modified with the logarithmic term. We therefore see that the superpotential generates a pure deformation flow, with vanishing vacuum expectation value for the operator $\langle\mathcal{O}_\Delta\rangle = 0$.

We would like to stress that the expansions (2.15) and (2.16) are slightly different from the usual Fefferman-Graham type expansions encountered in the literature [22, 24, 29]. In fact these are series in $z^{d-\Delta}$, whose exponent is in general *not* an integer. Apparently this fact has not been noticed before, since all flows extensively studied in the literature have integer exponents. The fact that there are no logarithmic terms is not in contradiction with the usual expansions since we are dealing with a Poincaré invariant background flow, for which those terms actually vanish.

[3]Notice that the linear coefficient in the expansion of e^A vanishes.

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3. The holographic c-function from scattering

In this section we will consider the holographic c-function proposed in [5] following field theoretical motivations. It can be computed as the Fourier transform of the transverse traceless (TT) stress-energy correlation function, related to the "flux"-factor initially discussed [6, 7] in the pure AdS/CFT case. In particular the c-function is related to the flux as

$$c(x) = (2\pi)^{\frac{d}{2}} x^{\frac{3}{2}d-3} \int_0^\infty dq\; q^{\frac{d}{2}-4} \mathcal{F}(q) J_{\frac{d}{2}-1}(qx)\,, \tag{3.1}$$

where $x = \sqrt{x_i x^i}$ measures distances in the boundary theory. The flux is obtained from the linearized fluctuations of the TT graviton in the background of an interpolating flow. By a slight redefinition of the metric fluctuations h_{ij}^{TT},

$$h_{ij}^{TT} = \mathrm{e}^{ikx}\,\chi(z)\,\xi_{ij}(x) \qquad \text{and} \qquad \chi(z) = \mathrm{e}^{-\frac{d-1}{2}A(z)}\,\psi(z)\,, \tag{3.2}$$

the linearized equation of motion can be casted [13] into a Schrödinger equation

$$-\frac{d^2}{dz^2}\,\psi(z) \;+\; \left[V_{\mathrm{QM}}(z) \;-\; k^2\right]\,\psi(z) = 0\,. \tag{3.3}$$

The quantum mechanical potential is of supersymmetric type and can be written as

$$V_{\mathrm{QM}} = W_{\mathrm{QM}}^2 + \frac{dW_{\mathrm{QM}}}{dz} \qquad \text{with} \qquad W_{\mathrm{QM}} \;\equiv\; \frac{d-1}{2}\frac{dA}{dz}\,. \tag{3.4}$$

There exists also a "supersymmetric partner potential" $\tilde{V}_{\mathrm{QM}}$, whose properties are closely related to V_{QM}, which is obtained simply by $W_{\mathrm{QM}} \to -W_{\mathrm{QM}}$. One can express the potentials in terms of the supergravity quantities by

$$V_{\mathrm{QM}} \;=\; -\,\frac{\mathrm{e}^{2A}}{16}\left(W^2 + 8V\right) \qquad \tilde{V}_{\mathrm{QM}} \;=\; \frac{\mathrm{e}^{2A}}{16}\left(3W^2 + 8V\right)\,. \tag{3.5}$$

These expressions do not depend on d and one can easily verify that $\tilde{V}_{\mathrm{QM}}$ is positive definite as long as $d > 2$.

We want to determine the type of quantum mechanical potential arising from an interpolating flow and focus on its local properties near the origin and infinity. The information about the potential in these regimes, together with the positivity of $\tilde{V}_{\mathrm{QM}}$ mentioned above, gives a handle to apply scattering theory to the problem in order to obtain some insight into the central function associated with the flow.

Substituting the expansions (2.15) and (2.16) into (3.5), the resulting expansion for the potential reads

$$V_{\mathrm{QM}}(z) = \frac{\lambda^2 - 1/4}{z^2} + \frac{1}{z^2}\left[v_2\,\tau^2 \;+\; \mathcal{O}(\tau^3)\right]\,, \tag{3.6}$$

where we have introduced $\lambda = d/2$. Notice that the leading term is independent of R and has the form of a centrifugal barrier in the radial Schrödinger equation for scattering off a central potential with angular momentum $l \;=\; (d-1)/2$. The "superpartner" $\tilde{V}_{\mathrm{QM}}$ has a

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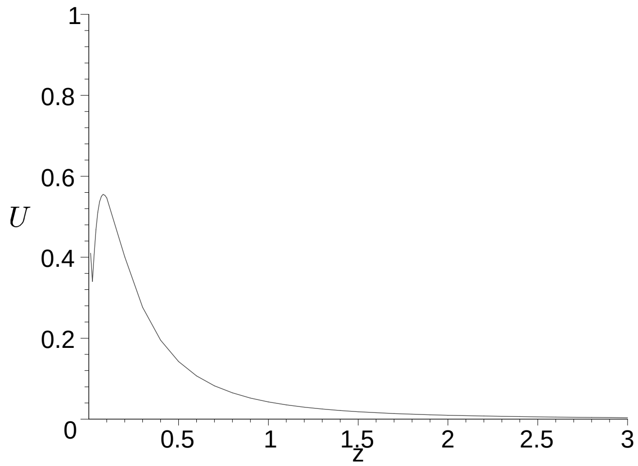

Figure 8: Central potentials associated with a flow with $\kappa_{\rm UV} > 2$.

similar expansion but with l replaced by $\tilde{l} = (d-3)/2$. The first non-zero coefficient of the expansion is given in the Appendix C. Since the centrifugal term in this expansion is the same in the UV and the IR limits, we split the potential in

$$V_{\rm QM}(z) \equiv \frac{\lambda^2 - 1/4}{z^2} \; + \; U(z) \tag{3.7}$$

and consider $U(z)$ as the central potential that generates the scattering of a particle in ordinary three dimensional space. From (3.6) it also follows that the leading behavior of $U(z)$ is given by

$$z \to 0 \quad : \qquad U(z) \;\sim\; \frac{1}{z^{2-\kappa_{\rm UV}}} \qquad 0 < \kappa_{\rm UV} \;=\; 2(d - \Delta_{\rm UV}) < d \tag{3.8}$$

$$z \to \infty \;\; : \qquad U(z) \;\sim\; \frac{1}{z^{2+\kappa_{\rm IR}}} \qquad 0 \;<\; \kappa_{\rm IR} \;=\; 2(\Delta_{\rm IR} - d) \tag{3.9}$$

Moreover $e^{2A(z)}$ is a non-singular, monotonically decreasing function, and $W^2 + 8V$ is bounded along the flow, so that $U(z)$ is an acceptable potential in scattering theory (see for instance [25]). In figure 8 and figure 9 we show the numerical results for the quantum mechanical potentials U and $\tilde{U}$ generated by the cubic potential of figure 1.

We now turn to the computation of the correlation function. The appropriate linear combination of independent solutions of the Schrödinger equation is selected by requiring regularity at the IR horizon. The solution exponentially suppressed in the Euclidean ($q = i|k|$) is the so-called Jost solution $\mathrm{f}(\lambda, -q, z)$, and using (A.4) this can be related to the non-singular and the singular solutions $\varphi(\pm\lambda, q, z)$, whose behavior near the origin is given by (A.2). If d is even (which is the case of interest) the singular solution contains a subleading logarithmic dependence on z, which must be taken into account in computing the correlation function.[4] In the notation of appendix A and by using the ϵ-prescription

[4]We must use the solutions of the eq. (3.3). As long as Δ is an *integer* the general theory of differential

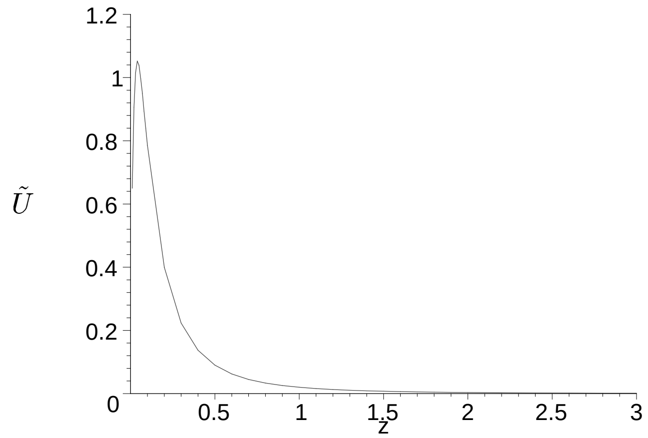

Figure 9: Central potentials associated with a flow with $\kappa_{\rm UV} > 2$.

for fixing the normalization [26] the solution we are interested in reads

$$\chi^{\epsilon}(q,z) = \left(\frac{\mathrm{e}^{-A(z)}}{\mathrm{e}^{-A(\epsilon)}}\right)^{\lambda-1/2} \frac{\mathcal{J}(\lambda,-q)\,\varphi(-\lambda,-q,z) - \mathcal{J}(-\lambda,-q)\,\varphi(\lambda,-q,z)}{\mathcal{J}(\lambda,-q)\,\varphi(-\lambda,-q,\epsilon) - \mathcal{J}(-\lambda,-q)\,\varphi(\lambda,-q,\epsilon)} \tag{3.11}$$

with $\mathcal{J}(\pm\lambda,-q)$ the Jost functions characteristic to the potential $U(z)$. This expression can now be used for computing the flux, as originally explained in [6, 7], and further developed in [27]–[30] for more general flows. We obtain the following expression

$$\mathcal{F}(q) = \mathcal{F}_{\log}(q) + \underbrace{-2\,\lambda\,R_{\rm UV}^{2\lambda-1}\,\frac{\mathcal{J}(-\lambda,-q)}{\mathcal{J}(\lambda,-q)}}_{\mathcal{F}_{\rm int}(q)}\,. \tag{3.12}$$

It should be mentioned that the first term is closely related to the behavior of the singular wave function at zero and contains in particular the contribution of the sub-leading logarithmic dependence. If U vanishes at the origin ($\kappa_{\rm UV} > 2$), the singular solution behaves as the $K_\lambda(q\,z)$ Bessel function and $\mathcal{F}_{\log}$ equals the pure AdS result. The second term ($\mathcal{F}_{\rm int}$) describes the interaction with the potential and is generic.

Notice that we have included the normalization given by the UV AdS radius, with the correct power (see e.g. [30] for a careful inclusion of the AdS radius in the correlation functions). While in IR-singular flows one can usually rescale the overall normalization as

equations tells us that the solutions defined at zero have the following structure

$$u_1(z) = z^{\lambda+1/2}\,f_1(z)\,, \qquad u_2(z) = z^{-\lambda+1/2}\,f_2(z) + \log(z)\,u_1(z)$$

with $f_1(z)$ and $f_2(z)$ analytic functions and the logarithmic contribution in the second solution appears because λ is an integer. If Δ is irrational, the above structure of the solutions is not valid any more, and one would have to analyze the problem on a case by case basis. However, if $d-\Delta = m/n$ with integers m,n (i.e. Δ is a rational number) by the change of variables $y = z^{1/n}$ one obtains an expansion of the above type, with $f_1(z)$ and $f_2(z)$ series in $z^{1/n}$, and the main arguments of the section go trough.

it is more convenient, this is strictly forbidden when considering interpolating flows, as in the IR limit an independent scale appears.

The Jost functions of the two quantum mechanical potentials $V_{\rm QM}$ and $\tilde{V}_{\rm QM}$ are in general related [31] as

$$\frac{\mathcal{J}(\lambda,k)}{\tilde{\mathcal{J}}(\lambda-1,k)} = \frac{k}{k-i\gamma_0}\,, \tag{3.13}$$

where γ_0^2 is the ground state energy of the potential $\tilde{V}_{\rm QM}$. This means that they have the same Jost functions, up to the addition of a bound state. In the present case $\tilde{V}_{\rm QM}$ is positive definite, hence it has no bound states and the continuum spectrum starts at zero energy. Therefore the Jost functions of the two potentials must be equal and in particular $\mathcal{J}(\lambda,-q)$ has no zeros [25]. It thus follows that in general $\mathcal{F}(q)$ has no massive poles in q, which means that the dual field theory does not have a mass gap or a spectrum of glue-balls, as expected from conformal invariance being restored in the IR.

In order to estimate the leading power terms of the c-function, we can extract its high and low energy behaviour by performing a Born type expansion of the flux factor $\mathcal{F}_{\rm int}$ in (3.12). As recalled in appendix A, the Jost functions admit a Born series expansion (cf. eq. (A.12) & eq. (A.13)) which follow from related integral equations,[5] Accordingly, one can expand the flux as $\mathcal{F}_{\rm int}(q) = \sum_{n=1}^{\infty}\mathcal{F}_n(q)$, where the first term reads

$$\begin{aligned}\mathcal{F}_1(q) &= -\,2\,\lambda\,R_{\rm UV}^{2\lambda-1}\,\frac{\mathcal{J}_1(-\lambda,-q)}{\mathcal{J}_0(\lambda,-q)}\\ &= -\,\frac{4\,\lambda}{\Gamma(\lambda)\,\Gamma(\lambda+1)}\,R_{\rm UV}^{2\lambda-1}\,\left(\frac{q}{2}\right)^{2\lambda}\int_0^{\infty} z\,U(z)\,K_\lambda(qz)^2\,dz\,.\end{aligned} \tag{3.14}$$

For large and small momenta q, the above integral is dominated by the contributions of the interaction potential $U(z)$ in the regions of $z\to 0$ and $z\to\infty$ respectively, where it behaves as in (3.8) and (3.9). Thus in these regimes we have

$$\mathcal{F}_1(q)\sim q^{2\Delta-d}\,, \tag{3.15}$$

with the relevant Δ in the two cases. This behavior is consistent with field theory expectations as can be seen by performing the integral transformation (3.1). In fact this gives for the leading contribution to the c-function $c(x)\sim x^{2(d-\Delta)}$. The results of [5, 32] state that generically in field theory the anomalous dimension h_* at the critical point of an RG flow is related to the c-function as

$$h_* = -\lim_{*}\frac{\ddot{c}}{\dot{c}}\,, \tag{3.16}$$

where the dot denotes the logarithmic derivative. Using this, we find

$$h_{\rm UV/IR} = \Delta_{\rm UV/IR} - d\,. \tag{3.17}$$

[5] We should mention that the relevant integral equation is singular near the origin. For instance, the integral in (3.14) is divergent at zero. However, assuming that ultimately one can extract the finite contribution, the behaviour in q is determined by a simple scaling argument.

This result was checked in the UV in [5, 9] for some specific examples. In fact we have seen that it should hold for any holographic flow near the UV and IR fixed points. To show that the c-function is monotonous along the flow and that the IR fixed point value is proportional to $R_{\rm IR}^{d-1}$ seems to involve a more detailed knowledge of the full potential $U(z)$. We will be able to check these properties in some toy model in the next section.

4. Models

In the previous section we have shown that the TT graviton fluctuation of an interpolating flow is equivalent to a scattering problem, with well-behaved QM potentials. In order to make further progress, we need to consider some specific model. We can deal with a feasible problem if we consider QM potentials that are piecewise solvable, and match them together at an arbitrary point. In the following we examine two models which correspond to δ-function and step barrier interactions. One can regard them as the most crude approximation to the generic potentials. Nevertheless, they turn out to encode most of the generic features expected from an interpolating flow.

4.1 Thin AdS/AdS wall

We construct the simplest interpolating "flow" by gluing together two AdS spaces with different radii. This is a thin-wall approximation adapted to the present problem. The resulting interaction potential is a δ-function at the gluing point.

In the z-coordinate system (the same conclusion can be obtained for instance in the r-coordinate) we consider two patches with local coordinates z_1 and z_2 and metrics

$$\mathrm{d}s_1^2 = \frac{R_{\rm UV}^2}{z_1^2}\left(\mathrm{d}z_1^2 + \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \qquad \mathrm{d}s_2^2 = \frac{R_{\rm IR}^2}{z_2^2}\left(\mathrm{d}z_2^2 + \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right). \tag{4.1}$$

Requiring that the metric on a constant z hypersurface be continuous at the gluing point, implies that $\bar{z}_1 R_{\rm IR} = \bar{z}_2 R_{\rm UV}$. In order to impose the correct boundary conditions for the

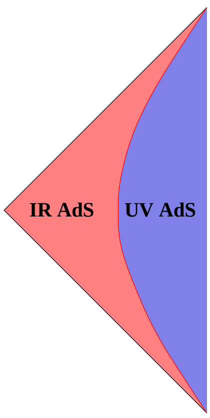

Figure 10: Thin AdS/AdS wall. Schematic diagrams of the models.

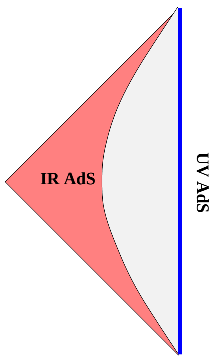

Figure 11: AdS/IK-space wall. Schematic diagrams of the models.

fluctuation at the matching point, it is more convenient to consider the physical fluctuation χ instead of the rescaled function ψ. The equation for χ reads

$$\chi''(z) \;+\; (d-1)A'(z)\chi'(z) - k^2\chi(z) = 0\,, \tag{4.2}$$

the solutions of which, in each of the two regions, are the free ones

$$\chi(z) = \left(\frac{z}{R}\right)^{\lambda}\left[a\,I_\lambda(q\,z) + b\,K_\lambda(q\,z)\right], \tag{4.3}$$

with $a = 0$ in the IR. The appropriate matching conditions on χ are

$$\chi_{\rm UV}(\bar z_1) = \chi_{\rm IR}(\bar z_2) \qquad \chi'_{\rm UV}(\bar z_1) = \chi'_{\rm IR}(\bar z_2)\,. \tag{4.4}$$

The first condition follows from requiring continuity of the metric, while the second one is a consequence of integrating (4.2) across the gluing surface.

The flux is now easily obtained, computing the two contributions to (3.12). In this case the potential vanishes at the origin, so that the logarithmic contribution is the pure UV fixed point term

$$\mathcal{F}_{\log}(q) \;=\; \underbrace{-\frac{4\lambda\,(-1)^\lambda}{\Gamma(\lambda)\,\Gamma(\lambda+1)}\,R_{\rm UV}^{2\lambda-1}\left(\frac{q}{2}\right)^{2\lambda}\log\,q}_{\mathcal{F}^{\rm UV}_{\rm CFT}(q)} \;+\; \mathcal{O}(q^{2\lambda})\;. \tag{4.5}$$

The non trivial part of the flux $\mathcal{F}_{\rm int}$ in eq. (3.12) reduces here to the quotient of $a_{\rm UV}$ and $b_{\rm UV}$ as follows

$$\mathcal{F}_{\rm int}(q) = -\frac{4\lambda}{\Gamma(\lambda)\,\Gamma(\lambda+1)}\,R_{\rm UV}^{2\lambda-1}\left(\frac{q}{2}\right)^{2\lambda}\frac{a_{\rm UV}}{b_{\rm UV}} \tag{4.6}$$

with

$$\frac{a_{\rm UV}}{b_{\rm UV}} = \frac{\Big[K_{\lambda+1}(q\bar z_1 r)K_\lambda(q\bar z_1) - K_\lambda(q\bar z_1 r)K_{\lambda+1}(q\bar z_1)\Big]q\bar z_1 r + d\,(r-1)\,K_\lambda(q\bar z_1 r)K_\lambda(q\bar z_1)}{\Big[K_{\lambda+1}(q\bar z_1 r)I_\lambda(q\bar z_1) + K_\lambda(q\bar z_1 r)I_{\lambda+1}(q\bar z_1)\Big]q\bar z_1 r + d\,(r-1)\,K_\lambda(q\bar z_1 r)I_\lambda(q\bar z_1)}\,, \tag{4.7}$$

where we have defined the quotient of the AdS radii $r = R_{\rm IR}/R_{\rm UV}$. From the above expression it is evident that $\mathcal{F}_{\rm int}$ vanishes if $R_{\rm IR} = R_{\rm UV}$, and one recovers the pure AdS result from (4.5). One can also, check using asymptotic expansions of the Bessel functions, that $\mathcal{F}_{\rm int}$ vanishes in the large q limit.

To check the IR behavior of the flux, we expand $a_{\rm UV}/b_{\rm UV}$ for small q

$$\frac{a_{\rm UV}}{b_{\rm UV}} = (-1)^\lambda\,\frac{R_{\rm IR}^{d-1} - R_{\rm UV}^{d-1}}{R_{\rm UV}^{d-1}}\,\log q + \frac{\Gamma(\lambda)\Gamma(\lambda-1)}{2}\,\frac{R_{\rm IR}-R_{\rm UV}}{R_{\rm UV}}\left(\frac{2}{q\bar z_1}\right)^{d-2} + \cdots\,, \tag{4.8}$$

where dots stand for higher order and contact terms. Notice that we have included the lowest order analytic term of the flux, which is quadratic in momentum and deserves a special attention. Inserting (4.8) into (3.12), we get the following small q expansion of the flux

$$\mathcal{F}(q) = \mathcal{F}^{\rm IR}_{\rm CFT}(q) + \frac{1}{2\,(\lambda-1)}\left(\frac{R_{\rm UV}}{\bar z_1}\right)^{d-2}(R_{\rm UV}-R_{\rm IR})\,q^2 \;+\cdots\,. \tag{4.9}$$

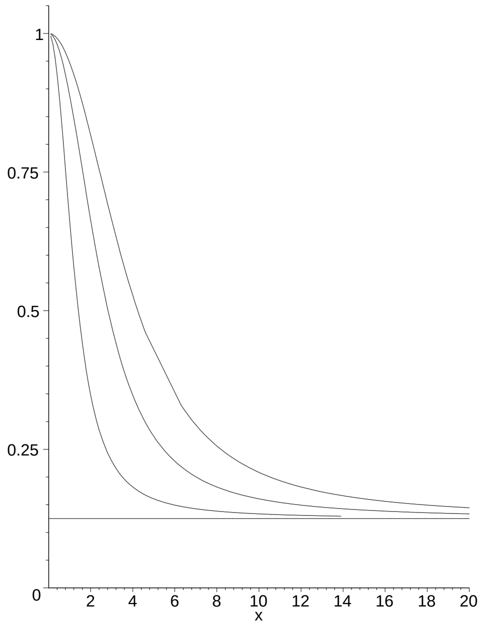

Figure 12: The c-function $\hat{c}(x)$ for $d = 4$, $R_{\mathrm{UV}} = 2$, $R_{\mathrm{IR}} = 1$ and $\bar{z}_1 = 1, 2, 3$.

Notice that the leading non-analytic term of eq. (4.6) has combined exactly with $\mathcal{F}^{\mathrm{UV}}_{\mathrm{CFT}}$ to reproduce the pure AdS IR result, independently of $\bar{z}_1$.

The last decisive test that we have to perform on the supposed c-function, is to check that it is in fact positive and monotonic. An analytical evaluation of the integral in (3.1) is not possible, therefore we have computed it numerically, for some specific values of the parameters. Note that the quadratic term in the small q expansion (4.9) is not a contact term and at large distances gives a contribution $\propto x^{d-2}$ in (3.1). An essential step for obtaining a meaningful central function is to subtract this contribution. Namely, we have computed the following "renormalized" c-function

$$\hat{c}(x) = c(x) - (4\pi)^{d/2} \frac{\Gamma(\lambda-1)}{8\,(\lambda-1)} \left(\frac{R_{\mathrm{UV}}}{\bar{z}_1}\right)^{d-2} (R_{\mathrm{UV}} - R_{\mathrm{IR}}) x^{d-2} \,. \tag{4.10}$$

A plot of $\hat{c}(x)$ is drawn in figure 12 below, where we have normalized $\hat{c}(0)$ to 1.

We see that $\hat{c}(x)$ obeys the properties expected of a central function, and in particular decreases towards $c_{\mathrm{IR}} > 0$ at large distances. This property has not been checked before in the context of holographic flows.

Let us comment on the issue of the quadratic term. In the context of holographic flows these terms have been discussed previously in [29]. In this reference it is pointed out that q^2-terms in the effective boundary action lead to massless poles in the correlation functions. Accordingly, it turns out that in the flow of [21] such terms cancel, which is consistent with the interpretation of an IR confining theory, with a mass-gap. On the other hand, a massless pole in a Coulomb branch flow is expected, as arising from the Goldstone boson associated with broken conformal invariance induced by the non-zero vev. It is more difficult to give a clear interpretation to our present model, since it does not arise as a smooth flow, and in particular there is no obvious description in terms of small perturbation of the UV CFT. Nonetheless, the resulting c-function is certainly suggestive of a non-trivial boundary theory, which has a UV *and* an IR CFT fixed points.

At the end of section 3 we have analyzed the small q behaviour of the flux stemming from an interaction potential characteristic of a smooth interpolating flow. We have seen that generically the first correction to the (logarithmic) fixed point behaviour goes like $q^{2\Delta_{\rm IR}-d}$. The analysis of section 3 relied on the expansions (2.15) (2.16) which are not valid for a delta-function potential, and as a result it gives rise to a spurious IR contribution which is quadratic. A generic long-range potential which arises from a smooth flow will prevent such quadratic terms.

4.2 AdS/IK-space domain wall

The previous model encodes most of the expected features related to the c-function of an interpolating flow. Nevertheless one might think that the result may lack generic features. Here we want to discuss an improved construction. Roughly speaking, we will give a finite thickness to the wall separating two different AdS regions, thus simulating a setting closer in spirit to an interpolating flow. In the language of the Schrödinger problem we are going to consider a step potential

$$U = \begin{cases} \omega^2 & \text{in patch } \{z_1\} \\ 0 & \text{in patch } \{z_2\} \end{cases} \tag{4.11}$$

combined with an additional δ-function located at the gluing point, i.e. we match to an AdS space (in the IR) with warp factor $A_{II}(z_2) = \log(R_{\rm IR}/z_2)$ a spacetime described by the a warp factor $A_I(z_1)$ obeying the equation

$$\left(\frac{d-1}{2}\right)^2 \left(\frac{dA_I}{dz_1}\right)^2 + \frac{d-1}{2}\frac{d^2A_I}{dz_1^2} = \frac{\lambda^2 - 1/4}{z_1^2} + \omega^2 \,. \tag{4.12}$$

The general solution to this equation is

$$A_I(z_1) = \frac{2}{d-1}\log\left[k_1\sqrt{z_1}\,K_\lambda(\omega z_1) + k_2\sqrt{z_1}\,I_\lambda(\omega z_1)\right], \tag{4.13}$$

where k_1 and k_2 are two positive constants, and we call the resulting spacetime an "IK-space", for obvious reasons. Given the warp factor for the metric, it is in principle straightforward to obtain the solution for $\phi(z)$ from (2.3), and eventually the supergravity potential V. However, for the computation of the c-function this information is not needed, and we have not performed this analysis here. In the limit $z_1 \to 0$ the warp factor asymptotes to

$$A_I(z_1) \to \log\left[\frac{R_{\rm UV}}{z_1}\right] \qquad \text{with} \qquad R_{\rm UV}^{2\lambda-1} = \left(\frac{2}{\omega}\right)^{2\lambda}\left(\frac{\Gamma(\lambda)}{2}\right)^2 k_1^2 \tag{4.14}$$

and so the space is asymptotically AdS. The solution of the fluctuation equation (4.2) in the patch $\{z_1\}$ is given by

$$\chi_I(z_1) = \frac{a_{\rm UV}\,I_\lambda(\epsilon\,z_1) + b_{\rm UV}\,K_\lambda(\epsilon\,z_1)}{k_1\,K_\lambda(\omega\,z_1) + k_2\,I_\lambda(\omega\,z_1)}\,, \qquad \epsilon = \sqrt{\omega^2+q^2} \tag{4.15}$$

and $\chi_{II}(z_2)$ is the free solution (4.3) used in the first model.

Notice that in principle one can choose the gluing point $\bar{z}_1$ such that $A_I(z_1)$ is monotonic, and tune the integration constant k_2 so to match the two regions smoothly, i.e. with continuous derivatives of the warp factors. We make here a simplifying assumption and set $k_2 = 0$. The requirement of continuity of the metric relates the two patches to each other via

$$\bar{z}_2 = \frac{R_{\rm IR}}{R_{\rm UV}} \left(\frac{2^{\lambda-1}\,\Gamma(\lambda)}{\omega^\lambda \sqrt{\bar{z}_1}\, K_\lambda(\omega\,\bar{z}_1)} \right)^{\frac{1}{\lambda-1/2}} . \tag{4.16}$$

For $\omega \to 0$ this approaches exactly the corresponding relation in the first example.

Requiring $\chi_{I/II}(z_{1/2})$ to be smooth as in (4.4), allows to determine the quotient of $a_{\rm UV}$ and $b_{\rm UV}$ as in the previous section. To shorten somewhat the expression, we have not written out explicitly the derivatives of Bessel functions, denoted with a prime

$$\frac{a_{\rm UV}}{b_{\rm UV}} = \frac{\left[\frac{d}{\bar{z}_2} K_\lambda(q\,\bar{z}_2) - q\,K_{\lambda+1}(q\,\bar{z}_2) \right] K_\lambda(\epsilon\,\bar{z}_1) - [K_\lambda(\epsilon\,\bar{z}_1)]' - K_\lambda(\epsilon\,\bar{z}_1)\,\frac{[K_\lambda(\omega\,\bar{z}_1)]'}{K_\lambda(\omega\,\bar{z}_1)}\,K_\lambda(q\,\bar{z}_2)}{\left[\frac{d}{\bar{z}_2} K_\lambda(q\,\bar{z}_2) - q\,K_{\lambda+1}(q\,\bar{z}_2) \right] I_\lambda(\epsilon\,\bar{z}_1) - [I_\lambda(\epsilon\,\bar{z}_1)]' - I_\lambda(\epsilon\,\bar{z}_1)\,\frac{[K_\lambda(\omega\,\bar{z}_1)]'}{K_\lambda(\omega\,\bar{z}_1)}\,K_\lambda(q\,\bar{z}_2)} \tag{4.17}$$

The quotient of Jost functions can be worked out and is proportional to the quotient of $a_{\rm UV}$ and $b_{\rm UV}$ as given below

$$\mathcal{F}_{\rm int}(q) = -\frac{4\lambda}{\Gamma(\lambda)\,\Gamma(\lambda+1)}\,R_{\rm UV}^{2\lambda-1} \left(\frac{\omega^2+q^2}{4} \right)^\lambda \frac{a_{\rm UV}}{b_{\rm UV}} . \tag{4.18}$$

In this case, however, the interaction potential does not vanish at the origin, hence the solutions (4.15) do not behave as the free Bessel solutions as mentioned in section 3. Now the contribution from the singular solution is not the pure AdS UV term and we obtain instead

$$\mathcal{F}_{\rm log}(q) = -\frac{4\lambda\,(-1)^\lambda}{\Gamma(\lambda)\,\Gamma(\lambda+1)}\,R_{\rm UV}^{2\lambda-1} \left(\frac{q^2+\omega^2}{4} \right)^\lambda \log\sqrt{q^2+\omega^2} + \mathcal{O}(q^0) . \tag{4.19}$$

This reduces to $\mathcal{F}_{\rm CFT}^{\rm UV}$ in the large q limit. However, since (4.19) is analytic at small q, the correct non-analytic term in the IR limit must follow from $\mathcal{F}_{\rm int}(q)$ alone. Making the expansion and utilizing some Bessel function identities, one can extract the leading non analytic contribution of (4.18) together with its leading quadratic term, written for $d = 4$ below

$$\begin{aligned} \mathcal{F}_{\rm int}(q) = & \underbrace{-\frac{1}{4}\,R_{\rm IR}^3\,q^4\,\log q}_{\mathcal{F}_{\rm CFT}^{\rm IR}(q)} + \\ & + R_{\rm UV}^3 \left[\frac{1}{8}\,\omega^4\,\bar{z}_1^2\,K_1(\omega\,\bar{z}_1)\,K_3(\omega\,\bar{z}_1) \; - \; \frac{1}{2}\,\frac{\bar{z}_1 + \bar{z}_2}{\bar{z}_2^3} \left(\frac{R_{\rm IR}}{R_{\rm UV}} \right)^3 \right] q^2 + \cdots . \end{aligned} \tag{4.20}$$

That we find the correct IR term is a non trivial check of the model and in agreement with the expectations. One can work out the quadratic terms, too. Now there are two sources of them, eq. (4.19) and eq. (4.20). In the limit $\omega \to 0$ the quadratic terms reproduce precisely the terms of the first example. We expect that after proper subtractions we should find a function $\hat{c}$ which interpolates the two central charges correctly. In this case the numerical evaluation of the integral in eq. (3.1) seems to be problematic and the corresponding plot is still missing.

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5. Conclusions

In this paper, we have addressed the issue of holographic interpolating flows. Contrary to Coulomb branch flows and IR-confining flows, in the literature there were no explicit examples of interpolating, i.e. IR-conformal, flows. Therefore the test of the AdS/CFT correspondence in these settings has been somewhat limited. In the first part we have characterized such flows in terms of their superpotentials and their expansions near the UV and IR fixed points. We have shown that the language of scattering theory is suitable for studying the fluctuations of the transverse traceless part of the metric, which are relevant for the computation of a holographic c-function.

Following these lines, we have proposed two explicit models which realize interpolating flows. The first model consists of a background space-time in which two AdS regions are glued together at an arbitrary distance from the boundary. In the second model we have replaced the outer AdS space (UV region) with an asymptotically AdS space. The corresponding scattering problems are solved explicitly.

In both cases we have shown that the c-function is non-trivial, and interpolates correctly between the UV and IR fixed point central charges. In particular, in our models the holographic c-function defined in terms of the superpotential [3] is trivial, and the related beta-function is ill-defined. On the other hand there is no problem to define a "proper" beta-function which is defined in term of the c-function [5, 33].

In the models we considered, we have found a quadratic term in the small momentum expansion of the flux-factor. We gave an argument that such a term will not be present in a smooth deformation flow. Therefore this feature should be an artefact of the toy models considered. Especially in the thin AdS/AdS domain wall, this should be related to the holographic image of a "brane" sitting in the bulk. It would be interesting to make this picture more clear.

Note added: A paper on a related subject, investigating a flow in two dimensional CFT, appeared on the arXive on the same day [34].

Acknowledgments

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A. Potential scattering

We follow reference [25]. The radial Schrödinger equation for the l-th partial wave for scattering off a central potential is

$$-\frac{d^2}{dz^2}\psi(z) + \left[\frac{l(l+1)}{z^2} + U(z) - k^2\right]\psi(z) = 0\,. \tag{A.1}$$

Let us define $\lambda = l + 1/2$. One can conveniently define two sets of independent solutions for the equation (A.1), which refer to boundary conditions given respectively at $z \to 0$ and $z \to \infty$. The "*regular*" solutions are defined by the boundary condition

$$\lim_{z \to 0} \varphi(\pm\lambda, k, z) z^{\mp\lambda - 1/2} = 1\,, \tag{A.2}$$

so that the non-singular and singular solutions are conveniently labeled by the sign in front of λ. The two solutions are always linearly independent (except for $\lambda = 0$). The *Jost solutions* are defined by the boundary condition

$$\lim_{z \to \infty} \mathrm{f}(\lambda, \pm k, z) \mathrm{e}^{\pm ikz} = 1\,. \tag{A.3}$$

For the application in mind we are interested in $\mathrm{f}(\lambda, -k, z)$, since it falls off exponentially at infinity upon Wick rotation to the positive imaginary axis. The two sets of solutions are related by

$$\begin{aligned} \mathrm{f}(\lambda, \pm k, z) &= \frac{1}{2\lambda} \left[\mathcal{J}(\lambda, \pm k) \varphi(-\lambda, \pm k, z) - \mathcal{J}(-\lambda, \pm k) \varphi(\lambda, \pm k, z) \right] &\text{(A.4)} \\ \varphi(\pm\lambda, k, z) &= \frac{1}{2ik} \left[\mathcal{J}(\pm\lambda, k) \mathrm{f}(\lambda, -k, z) - \mathcal{J}(\pm\lambda, -k) \mathrm{f}(\lambda, k, z) \right]. &\text{(A.5)} \end{aligned}$$

The coefficients $\mathcal{J}(\pm\lambda, \pm k)$ are called *Jost functions* and can be obtained by suitable wronskians, e.g.

$$\mathcal{J}(\lambda, k) \equiv W[\mathrm{f}(\lambda, k, z), \varphi(\lambda, k, z)]\,. \tag{A.6}$$

They are independent of z, since the wronskian of any two solutions of (A.1) is a constant. We now specialize to Euclidean momenta $q = i|k|$. The regular solutions and the Jost solutions for the free case, i.e. $U(z) \equiv 0$, read respectively

$$\begin{aligned} \varphi_0(+\lambda, q, z) &= 2^\lambda\, \Gamma(\lambda + 1)\, q^{-\lambda} \sqrt{z}\, I_\lambda(qz)\,, & f_0(\lambda, -q, z) &= \sqrt{2qz/\pi}\, K_\lambda(qz) &\text{(A.7)} \\ \varphi_0(-\lambda, q, z) &= \frac{2^{1-\lambda}\, q^\lambda}{\Gamma(\lambda)} \sqrt{z}\, K_\lambda(qz)\,, & f_0(\lambda, +q, z) &= \sqrt{2\,\pi\, qz}\, I_\lambda(qz) &\text{(A.8)} \end{aligned}$$

and the free Jost functions are

$$\mathcal{J}_0(\lambda, -q) = \sqrt{2/\pi}\, 2^\lambda\, \Gamma(\lambda + 1)\, q^{-\lambda + 1/2}\,, \qquad \mathcal{J}_0(-\lambda, -q) = 0\,. \tag{A.9}$$

The Jost solution can be constructed recursively, i.e. $f(\lambda, \pm q, z) = \sum_{n=0}^{\infty} f_n(\lambda, \pm q, z)$. Each term in this expansion can be obtained by

$$f_n(\lambda, \pm q, z) = \int_z^\infty B(\lambda, q, z, \xi)\, U(\xi)\, f_{n-1}(\lambda, \pm q, \xi)\, d\xi\,. \tag{A.10}$$

Here the kernel is $B(\lambda, q, z, \xi) = \sqrt{z\xi}\, [\, K_\lambda(q\xi)\, I_\lambda(qz) - K_\lambda(qz)\, I_\lambda(q\xi)\,]$. The integral representation for the Jost function reads

$$\mathcal{J}(\pm\lambda, -q) = \mathcal{J}_0(\pm\lambda, -q) + \int_0^\infty \varphi_0(\pm\lambda, q, \xi) U(\xi) f(\lambda, -q, \xi) d\xi\,. \tag{A.11}$$

These expressions are useful for deriving Born type expansions. In particular from eq. (A.11) we obtain an expansion of the Jost function

$$\mathcal{J}(-\lambda,-q)=\sum_{n=0}^{\infty}\mathcal{J}_n(-\lambda,-q)\,, \tag{A.12}$$

whose first order correction reads

$$\mathcal{J}_1(-\lambda,-q)=\sqrt{2/\pi}\,\frac{2^{-\lambda}}{\Gamma(\lambda)}\,q^{\lambda+1/2}\int_0^{\infty}\xi\,U(\xi)\,K_\lambda^2(q\xi)\,d\xi\,. \tag{A.13}$$

We refer the reader to [25] for further details.

B. Counterterms and anomalies

In this appendix we make some comments on the expansions of the superpotentials of section 2.2 in relation to the issue of "holographic renormalization" [24, 20, 29].

In general the on-shell action, which is used for computing holographic correlation functions [6, 7], suffers from divergences arising integrating certain terms near the AdS boundary. These divergences can be isolated and subtracted, in order to obtain a finite renormalized on-shell action. Some of these terms may be determined by analyzing the simple background solutions, though more general solutions (x-dependent) must be considered to find all of the counterterms. Here we focus our attention on the former type, so that all possible divergent terms are contained in the boundary term

$$S=\int_{z=\epsilon}d^dx\sqrt{\hat{g}}W(\phi)\,, \tag{B.1}$$

where ϵ is a (UV or IR) cut-off, and $\hat{g}_{ij}$ is the induced metric. Given the expansions (2.15) and (2.16), for a generic background flow near a fixed point, divergences can arise from the leading terms in the expansion[6]

$$\sqrt{\hat{g}}W(\Phi)=\frac{1}{\epsilon^d}\left(W_0+\frac{1}{2}W_0^{(2)}\epsilon^{2(d-\Delta)}+\cdots\right). \tag{B.2}$$

First, let us rule out the possibility that divergences (hence anomalies) arise near the IR fixed point. According to (2.9), the exponents $n(d-\Delta)-d$ are negative, and therefore there are no divergences as the IR region is reached at $\epsilon\to\infty$.

Near the UV boundary, the divergent terms are those of order $n<[d/(d-\Delta)]$. A finite term can arise only for operators saturating the above bound. We have seen that they give rise to the logarithmic-modified expansion for W (2.12). In turn, the expansions (2.15) and (2.16) become more complicated. However, at leading order they are unchanged, so that the divergent part of (B.1) can be read off as

$$\left[\sqrt{\hat{g}}W(\Phi)\right]_{\mathrm{div}}=\frac{1}{\epsilon^d}W_0+\cdots+\frac{W_0^{(n)}}{n!}+(d-\Delta)\frac{X_0^{(n)}}{n!}\log\epsilon\,. \tag{B.3}$$

[6]For compactness of notation in this Appendix we have set $R=1$.

The first $n-1$ terms are divergent and must be subtracted from the on-shell action. We have seen that the finite term was left undetermined by the equations of motion, and here we see explicitly that it corresponds to a choice of renormalization scheme. Furthermore we can now interpret the logarithmic term. Upon varying the on-shell action this contributes to the fixed point conformal anomaly via the mechanism of [35]. Thus, as expected [24, 23], we get the following contribution to the Weyl anomaly due to scalar operators of dimensions as in (2.13)

$$\mathcal{A}_{\mathrm{matter}} = \frac{d}{(n+1)!} X_0^{(n)} \Phi^n \,. \tag{B.4}$$

Interestingly, it encodes information about the gravity potential V, up to order n.

C. Coefficients of the asymptotic expansions

Here we list few coefficients in the expansions of ϕ and e^A as indicated in eq. (2.15)

$$\begin{aligned} c_1 &= \frac{R}{2} \frac{W_0^{(3)}}{\Delta - d} \\ c_2 &= \frac{R^2}{4} \frac{W_0^{(3)2}}{(\Delta-d)^2} - \frac{R}{12} \frac{W_0^{(4)}}{(\Delta-d)} + \frac{1}{8} \frac{(\Delta-d)}{[\,2\,(\Delta-d)-1\,]\,(d-1)} \end{aligned}$$

and eq. (2.16)

$$\begin{aligned} a_2 &= -\frac{1}{4} \frac{(\Delta-d)}{(d-1)\,[\,2\,(\Delta-d)\,-\,1\,]} \\ a_3 &= -\frac{1}{3} \frac{W_0^{(3)}\,R}{(d-1)\,[\,3\,(\Delta-d)\,-\,1\,]} \end{aligned}$$

with

$$\begin{aligned} W_0^{(3)} &= \frac{R}{3\,\Delta\,-\,2\,d} V_0^{(3)} \\ W_0^{(4)} &= \frac{R}{4\,\Delta-3\,d} \left[V_0^{(4)} + \frac{3\,d\,(\Delta-d)^2}{2\,R^2\,(d-1)} - \frac{3\,R^2\,V_0^{(3)2}}{(3\Delta-2d)^2} \right]. \end{aligned}$$

The coefficients in the expansion of V_{QM} in eq. (3.5) follow immediately and the first nonzero one reads

$$v_2 = -\frac{1}{4} \frac{(\Delta-d)^2}{2\,(\Delta-d)\,-\,1} [\,2\,\Delta - d\,].$$

In principle these expansions are valid in the UV and the IR. However, in the IR, due to the bound of eq. (2.9) some of the denominators can vanish, making the expansion not valid. This happens for operators of IR dimension $\Delta_{\mathrm{IR}} = d + 1/n$, for some integer n. In the UV the sign of v_2 is positive due to the bound of eq. (2.8). The first coefficient of the expansion of $\tilde{V}_{\mathrm{QM}}$ in eq. (3.5) is

$$\tilde{v}_2 = \frac{1}{4} \frac{(\Delta-d)^2}{2\,(\Delta-d)-1} [\,2\,\Delta - d \,+\, 2(1-d)\,] \,.$$

References

[1] A.B. Zamolodchikov, *'Irreversibility' of the flux of the renormalization group in a 2-d field theory*, *Sov. Phys. JETP Lett.* **43** (1986) 730 [*Zh. Eksp. Teor. Fiz.* **43** (1986) 565].

[2] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, *J. High Energy Phys.* **12** (1998) 022 [hep-th/9810126].

[3] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017].

[4] D. Anselmi, *Central functions and their physical implications*, *J. High Energy Phys.* **05** (1998) 005 [hep-th/9702056].

[5] D. Anselmi, L. Girardello, M. Porrati and A. Zaffaroni, *A note on the holographic beta and c functions*, *Phys. Lett.* **B 481** (2000) 346 [hep-th/0002066].

[6] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[7] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[8] M. Porrati and A. Starinets, *Holographic duals of 4d field theories*, hep-th/0009198.

[9] M. Porrati and A. Starinets, *On the canonical c-function in 4-d field theories possessing supergravity duals*, *Phys. Lett.* **B 498** (2001) 285 [hep-th/0009227].

[10] J. Distler and F. Zamora, *Non-supersymmetric conformal field theories from stable anti-de Sitter spaces*, *Adv. Theor. Math. Phys.* **2** (1999) 1405 [hep-th/9810206].

[11] O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, *Modeling the fifth dimension with scalars and gravity*, *Phys. Rev.* **D 62** (2000) 046008 [hep-th/9909134].

[12] V.L. Campos, G. Ferretti, H. Larsson, D. Martelli and B.E.W. Nilsson, *A study of holographic renormalization group flows in $D = 6$ and $D = 3$*, *J. High Energy Phys.* **06** (2000) 023 [hep-th/0003151].

[13] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, *J. High Energy Phys.* **07** (2000) 038 [hep-th/9906194].

[14] M. Porrati and A. Starinets, *RG fixed points in supergravity duals of 4-d field theory and asymptotically AdS spaces*, *Phys. Lett.* **B 454** (1999) 77 [hep-th/9903085].

[15] M. Cvetič, S.S. Gubser, H. Lu and C.N. Pope, *Symmetric potentials of gauged supergravities in diverse dimensions and Coulomb branch of gauge theories*, *Phys. Rev.* **D 62** (2000) 086003 [hep-th/9909121].

[16] I. Bakas, A. Brandhuber and K. Sfetsos, *Domain walls of gauged supergravity, m-branes and algebraic curves*, *Adv. Theor. Math. Phys.* **3** (1999) 1657 [hep-th/9912132].

[17] K. Skenderis and P.K. Townsend, *Gravitational stability and renormalization-group flow*, *Phys. Lett.* **B 468** (1999) 46 [hep-th/9909070].

[18] P. Breitenlohner and D.Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett.* **B 115** (1982) 197.

JHEP04(2002)27

[19] P.K. Townsend, *Positive energy and the scalar potential in higher dimensional (super)gravity theories*, *Phys. Lett.* **B 148** (1984) 55.

[20] J. Kalkkinen, D. Martelli and W. Muck, *Holographic renormalisation and anomalies*, *J. High Energy Phys.* **04** (2001) 036 [hep-th/0103111].

[21] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *The supergravity dual of $N = 1$ super Yang-Mills theory*, *Nucl. Phys.* **B 569** (2000) 451 [hep-th/9909047].

[22] C. Fefferman and C.R. Graham, *Conformal invariants*, in *Elie Cartan et les mathématiques d'ajourd'hui*, Astérisque, 1985.

[23] A. Petkou and K. Skenderis, *A non-renormalization theorem for conformal anomalies*, *Nucl. Phys.* **B 561** (1999) 100 [hep-th/9906030].

[24] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230].

[25] V. De Alfaro and T. Regge, *Potential scattering*, North-Holland, Amsterdam 1965.

[26] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT_d/AdS_{d+1} correspondence*, *Nucl. Phys.* **B 546** (1999) 96 [hep-th/9804058].

[27] O. DeWolfe and D.Z. Freedman, *Notes on fluctuations and correlation functions in holographic renormalization group flows*, hep-th/0002226.

[28] G. Arutyunov, S. Frolov and S. Theisen, *A note on gravity-scalar fluctuations in holographic RG flow geometries*, *Phys. Lett.* **B 484** (2000) 295 [hep-th/0003116].

[29] M. Bianchi, D.Z. Freedman and K. Skenderis, *How to go with an RG flow*, *J. High Energy Phys.* **08** (2001) 041 [hep-th/0105276].

[30] W. Mück, *Correlation functions in holographic renormalization group flows*, *Nucl. Phys.* **B 620** (2002) 477 [hep-th/0105270].

[31] B. Talukdar, K. Roy, C. Bhattacharyya and U. Das, *Jost functions in supersymmetric quantum mechanics*, *Phys. Lett.* **A 136** (1989) 93.

[32] D. Anselmi, M.T. Grisaru and A. Johansen, *A critical behaviour of anomalous currents, electric-magnetic universality and CFT_4*, *Nucl. Phys.* **B 491** (1997) 221 [hep-th/9601023].

[33] D. Anselmi, *Anomalies, unitarity and quantum irreversibility*, *Ann. Phys. (NY)* **276** (1999) 361 [hep-th/9903059].

[34] M. Berg and H. Samtleben, *An exact holographic RG flow between 2d conformal fixed points*, hep-th/0112154.

[35] M. Henningson and K. Skenderis, *The holographic weyl anomaly*, *J. High Energy Phys.* **07** (1998) 023 [hep-th/9806087].

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